Heisenberg-Robertson Generalized Uncertainty Relation

Consider two Hermitian observables $\hat{A}$ and $\hat{B}$. Given a state $|\psi\rangle$, the expected values of a von Neumann measurement are $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$, $\langle B\rangle = \langle\psi|\hat{B}|\psi\rangle$, the the "uncertainty" in the measurement outcome is $\Delta A = \sqrt{\langle\Delta\hat{A}^2\rangle}$, $\Delta B = \sqrt{\langle\Delta\hat{B}^2\rangle}$

where $\Delta\hat{A} = \hat{A} - \langle A\rangle$, $\Delta\hat{B} = \hat{B} - \langle B\rangle$

Consider the "mean-squared error" (variance)

$$\langle\Delta\hat{A}^2\rangle = \langle\psi|(\hat{A}-\langle A\rangle)(\hat{A}-\langle A\rangle)|\psi\rangle = \langle\psi_{\Delta A}|\psi_{\Delta A}\rangle = \|\psi_{\Delta A}\|^2, \text{ where } |\psi_{\Delta A}\rangle = (\hat{A}-\langle A\rangle)|\psi\rangle$$

Similarly: $\langle\Delta\hat{B}^2\rangle = \langle\psi_{\Delta B}|\psi_{\Delta B}\rangle = \|\psi_{\Delta B}\|^2$ where $|\psi_{\Delta B}\rangle = (\hat{B}-\langle B\rangle)|\psi\rangle$

Consider then $\langle\psi_{\Delta A}|\psi_{\Delta B}\rangle = \langle\psi|\Delta\hat{A}\,\Delta\hat{B}|\psi\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$

Similarly $\langle\psi_{\Delta B}|\psi_{\Delta A}\rangle = \langle\hat{B}\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$

According to Cauchy-Schwarz: $|\langle\psi_{\Delta A}|\psi_{\Delta B}\rangle|^2 \leq \|\psi_{\Delta A}\|^2\|\psi_{\Delta B}\|^2 = \Delta A^2\Delta B^2$

And $|\langle\psi_{\Delta A}|\psi_{\Delta B}\rangle|^2 \geq \left[\mathrm{Im}(\langle\psi_{\Delta A}|\psi_{\Delta B}\rangle)\right]^2 = \left[\dfrac{\langle\psi_{\Delta A}|\psi_{\Delta B}\rangle - \langle\psi_{\Delta B}|\psi_{\Delta A}\rangle}{2i}\right]^2$

$$|\langle\psi_{\Delta A}|\psi_{\Delta B}\rangle|^2 \geq \left[\frac{\langle\hat{A}\hat{B} - \hat{B}\hat{A}\rangle}{2i}\right]^2 = \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2$$

$\therefore$ $\boxed{\Delta A\,\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|}$ Generalized uncertainty relation

For the specific case $\hat{A} = \hat{x}$, $\hat{B} = \hat{p}$, $[\hat{A},\hat{B}] = i\hbar$

$\Rightarrow$ $\boxed{\Delta x\,\Delta p \geq \frac{\hbar}{2}}$ The famous Heisenberg uncertainty relation.